# Econ 802

## Answers to Second Midterm

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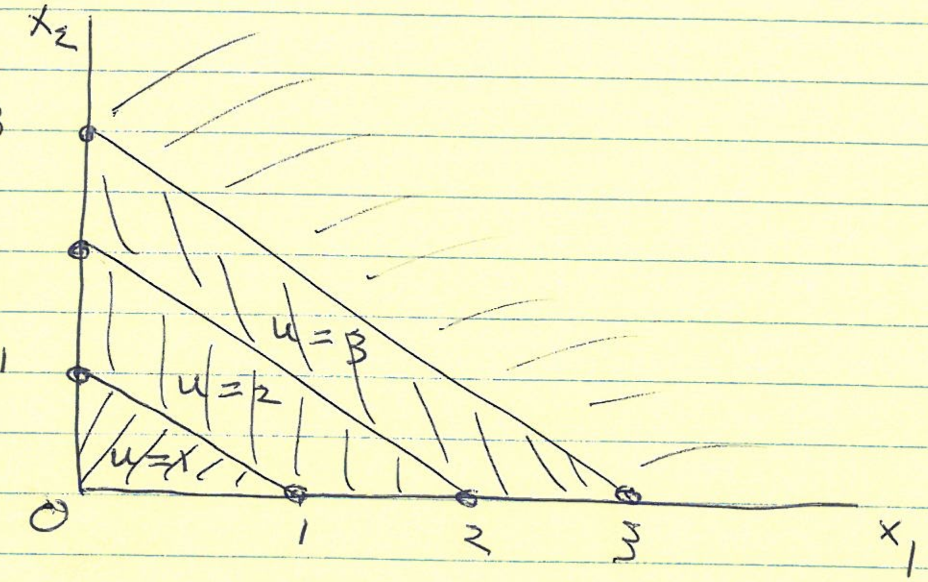
1. (a)




All bundles above (0,0) but on or below $x_1 + x_2 = 1$ are indifferent to each other. Likewise, all bundles above $x_1 + x_2 = 1$ but on or below $x_1 + x_2 = 2$ are indifferent to each other and so on. The upper contour set for $u = 1$ is the set of all points above the origin; the upper contour set for $u = 2$ is the set of all points above $x_1 + x_2 = 1$, and so on.

(b) Weak monotonicity: <u>yes</u>. If $x \geq y$ then $u(x) \geq u(y)$.

Strong monotonicity: <u>no</u>. If we start from a point $y$ in the <u>interior</u> of the indifference set for $u = 2$, for example, we can add small amounts of both goods and get $u(x) = u(y)$.

Strict convexity: <u>no</u>. We can choose bundles $x \neq y$ and $z$ all in the indifference set for $u = 2$ and any convex combination of $(x, y)$ will be indifferent to $z$.

(c) There is always a solution for any $(p, m) > 0$. The budget set $B = \{x \geq 0 : px \leq m\}$ is non-empty, closed and bounded. Because $x_1 + x_2$ is a continuous function, there is some point in $B$ that maximizes it. Call this point $x^*$. Every other $x \in B$ has $x_1 + x_2 \leq x_1^* + x_2^*$ and therefore has $u(x) \leq u(x^*)$. Therefore $x^*$ maximizes utility at $(p, m)$.

2. (a) We need to solve $\max\ x_1^{1/2} + x_2$ subject to

$$p_1 x_1 + p_2 x_2 = m.$$

From the budget constraint, $x_2 = \dfrac{m - p_1 x_1}{p_2}$

Substitute this into the utility function:

$$\Rightarrow \max\ x_1^{1/2} + \frac{m}{p_2} - \frac{p_1 x_1}{p_2}$$

FOC: $\frac{1}{2} x_1^{-1/2} - \frac{p_1}{p_2} = 0$

SOC: $(\frac{1}{2})(-\frac{1}{2}) x_1^{-3/2} < 0$ (This is sufficient)

Marshallian demand for good 1: $x_1 = \left(\dfrac{p_2}{2p_1}\right)^2$

Substitute this into budget constraint to get

$$x_2 = \frac{m}{p_2} - \frac{1}{4}\left(\frac{p_2}{p_1}\right)$$

Indirect utility: $v(p, m) = \dfrac{p_2}{2p_1} + \dfrac{m}{p_2} - \dfrac{1}{4}\left(\dfrac{p_2}{p_1}\right)$

or $v(p, m) = \dfrac{p_2}{4p_1} + \dfrac{m}{p_2}$

(b) We need to solve $\min\ p_1 x_1 + p_2 x_2$ subject to

$$x_1^{1/2} + x_2 = u$$

From the utility constraint, $x_2 = u - x_1^{1/2}$

Substitute this into the expenditure expression $\Rightarrow$

$$\min \; p_1 x_1 + p_2 u - p_2 x_1^{1/2}$$

FOC: $p_1 - p_2\left(\tfrac{1}{2}\right)x_1^{-1/2} = 0$

SOC: $\left(\tfrac{-1}{2}\right)^2 p_2 x_1^{-3/2} > 0$ (this is sufficient)

Hicksian demand for good 1: $h_1(p,u) = \left(\frac{p_2}{2p_1}\right)^2$

Substitute this into utility constraint to get

$$h_2(p,u) = u - \frac{p_2}{2p_1}.$$

Expenditure function: $e(p,u) = p_1\left(\frac{p_2}{2p_1}\right)^2 + p_2\left[u - \frac{p_2}{2p_1}\right]$

or $e(p,u) = p_2 u - \frac{p_2^2}{4p_1}$

(c) Note that the Marshallian demand for good 1 is equal to the Hicksian demand for good 1:

$$x_1(p,m) = \left(\frac{p_2}{2p_1}\right)^2 = h_1(p,u)$$

This is unusual because normally income ($m$) would appear in the Marshallian demand function and utility ($u$) would appear in the Hicksian function.

The reason for the unusual result is that this utility function is quasi-linear. As a consequence, there are no income effects for good 1. For the same reason, there are no effects from the utility level in the Hicksian demand.

This is a special case of a more general fact: when utility is quasi-linear, the Marshallian and Hicksian demands are identical for the good having <u>non-linear</u> utility, and demand for this good only depends on the price ratio.

3. (a) Writing the Slutsky equation in matrix form,

$$\underset{(n \times n)}{\frac{\partial x(p,m)}{\partial p}} = \underset{(n \times n)}{\frac{\partial h[p, v(p,m)]}{\partial p}} - \underset{(n \times 1)}{\frac{\partial x(p,m)}{\partial m}} \cdot \underset{(1 \times n)}{x(p,m)}$$

Rearranging this: $\frac{\partial h[p, v(p,m)]}{\partial p} = \frac{\partial x(p,m)}{\partial p} + \frac{\partial x(p,m)}{\partial m} x(p,m)$

From Shepherd's Lemma, the Hicksian demands are the derivatives of the expenditure function:

$$\left[\frac{\partial e(p,u)}{\partial p_1}, \ldots \frac{\partial e(p,u)}{\partial p_n}\right] = \left[h_1(p,u) \ldots h_n(p,u)\right]$$

Therefore the Hessian matrix of the expenditure function is equal to the substitution matrix:

$$\frac{\partial^2 e(p,u)}{\partial p^2} = \frac{\partial h(p,u)}{\partial p}$$

Recalling that the expenditure function is concave we know that $\frac{\partial h(p,u)}{\partial p}$ is negative semi-definite.

Therefore from the Slutsky equation,

$\frac{\partial x(p,m)}{\partial p} + \frac{\partial x(p,m)}{\partial m} x(p,m)$ is also negative semi-def.

Because $(p,m)$ and $x$ are observable (at least potentially), this places a number of restrictions on the Marshallian demand functions $x(p,m)$ and in principle we could test these hypotheses using econometric methods.

3(b) The student is ignoring the role of time endowments in the budget constraint. Suppose the consumer solves

$$\max\ u(c, L) \text{ subject to } pc \leq w(T-L)$$

$$\text{or } pc + wL \leq wT$$

where $c$ = consumption
$T$ = total time
$L$ = leisure
$p$ = price, $w$ = wage.

The Marshallian demand for leisure is $L(p, w, m)$ where we substitute $m = wT$ to get $L(p, w, wT)$. When the wage increases, two things happen: the price of leisure goes up, which the student correctly says will lead to substitution away from leisure toward more work hours; and also income $wT$ goes up. If leisure is a normal good, the second effect may outweigh the first one, and the consumer demands more leisure. If so, labor supply $T-L$ goes down.

(c) To justify putting "all other goods" on the vertical axis, we have to justify aggregating those goods and treating them as a composite commodity. One approach is Hicksian separability. Let $x$ be a single good (on the horizontal axis) and let $z$ be a vector of all other goods. Let the prices for the $z$ goods be $q = tq_0$ where $q_0$ is fixed and $t$ is a scalar. Define the composite commodity $Z \equiv q_0 z$. Write $v(p, t, m) = \max u(x, z)$ subject to $px + tq_0 z = m$ and then define the new direct utility function

$w(x, Z) = \min v(p, t, 1)$ subject to $px + tZ = 1$.
We can work with the new direct utility function which only has two arguments, ($x$ and the composite commodity $Z$), where we use $t$ as the price of $Z$. This allows us to draw indifference curves for two goods in the usual way, where the Marshallian demand for $x$ will be $x(p, t, m)$ or $x(\frac{p}{t}, 1, \frac{m}{t})$ if we use the zero degree homogeneity of the function.

4(a)

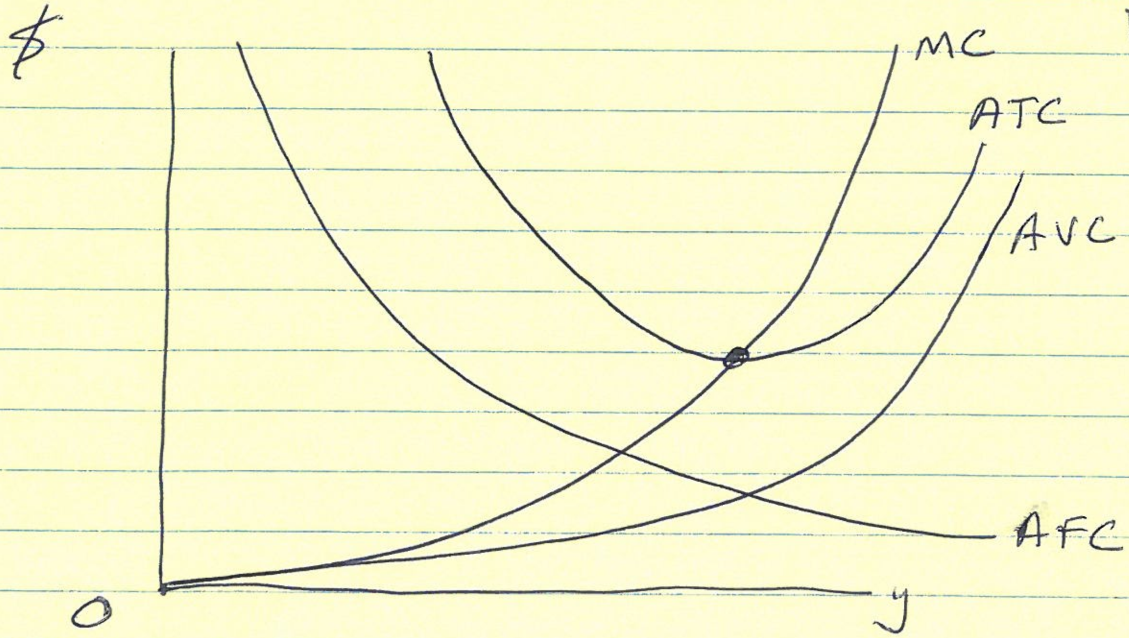
$$AFC = \frac{rK}{y}$$

$$AVC = w y^{\frac{1}{\beta}-1} K^{-\frac{\alpha}{\beta}}$$

$$ATC = \frac{rK}{y} + w y^{\frac{1}{\beta}-1} K^{-\frac{\alpha}{\beta}}$$

$$MC = \frac{w}{\beta} y^{\frac{1}{\beta}-1} K^{-\frac{\alpha}{\beta}}$$

<u>Note</u>: (1) The slope of MC is increasing due to $\beta < 1/2$.
(2) MC > AVC due to $\beta < 1$
(3) ATC is U-shaped due to $\beta < 1/2$
(4) MC passes through min of ATC.



4(b) Profit max by firm $i$ gives the FOC

$$p = MC_i \quad \text{or} \quad p = \frac{w}{\beta} y_i^{\frac{1}{\beta}-1} K_i^{-\frac{\alpha}{\beta}}, \quad i = 1..n$$

Note: SOC holds because $MC_i$ is rising.

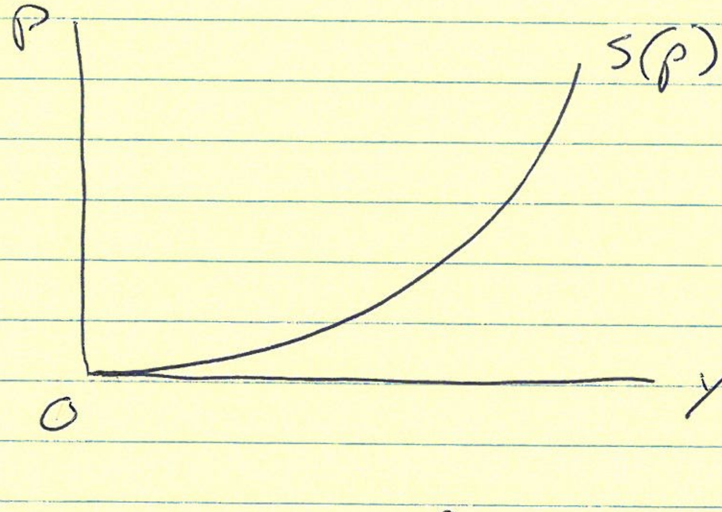
Because $MC_i(y_i) > AVC_i(y_i)$ for all $y_i > 0$, there is no shutdown issue (it is always true that $p > AVC_i$ when FOC holds).

To get firm $i$'s supply function, solve for $y_i$:

$$y_i(p) = \left(\frac{p\beta}{w}\right)^{\frac{\beta}{1-\beta}} K_i^{\frac{\alpha}{1-\beta}}$$

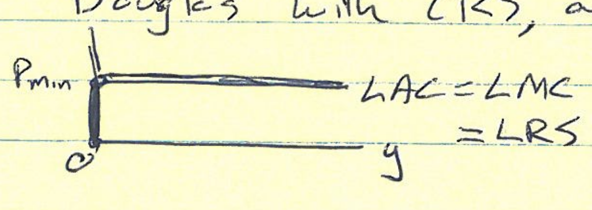
The market supply function is

$$S(p) = \sum_{i=1}^{n} y_i(p) = \left(\frac{p\beta}{w}\right)^{\frac{\beta}{1-\beta}} \sum_{i=1}^{n} K_i^{\frac{\alpha}{1-\beta}}$$




Note: The slope of $S(p)$ is increasing due to $\beta < 1/2$ (The same reason why $MC_i$ has an increasing slope for each $i$)

(c) To get some information about long run supply, we would like to know the production function. Notice from the short run cost function that labor input is $L = y^{1/\beta} K^{-\alpha/\beta}$. Rearranging this gives $y = K^\alpha L^\beta$ where $\alpha + \beta = 1$. So this is Cobb-Douglas with CRS, and LAC is horizontal:




The long run market supply curve is vertical at zero for $p \leq p_{min}$ and horizontal at $p_{min}$. (undefined for $p > p_{min}$)

5(a) Each consumer solves the problem

$$\max \; a x_i - b x_i^2 + y_i$$

subject to $\; p x_i + y_i = w$

$\Rightarrow$

$$\max \; a x_i - b x_i^2 + w - p x_i$$

$\Rightarrow$ FOC is $\; a - 2 b x_i - p = 0$ $\Big\}$ note: SOC is automatic

$$\Rightarrow \quad x_i = \frac{a-p}{2b}$$

There are n identical consumers, so market demand is $D(p) = \dfrac{n(a-p)}{2b}$ for $p \le a$; $D(p) = 0$ for $p \ge a$.

Each firm solves the problem

$$\max \; p z_j - c_j(z_j)$$

or $\; \max \; p z_j - g z_j - h z_j^2$

$\Rightarrow$ FOC is $\; p - g - 2 h z_j = 0$

$$\Rightarrow \quad z_j = \frac{p-g}{2h}$$

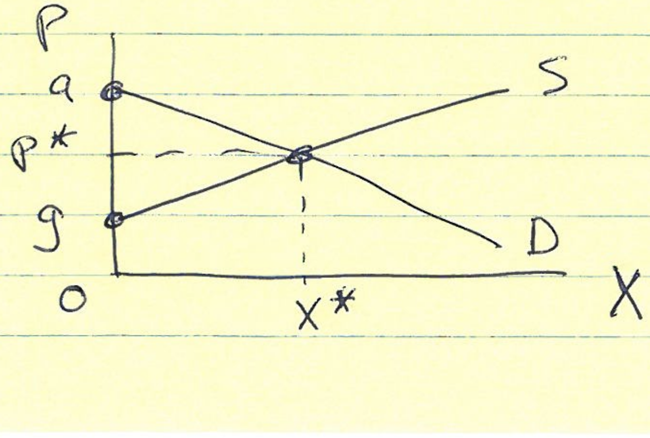
There are m identical firms, so market supply is

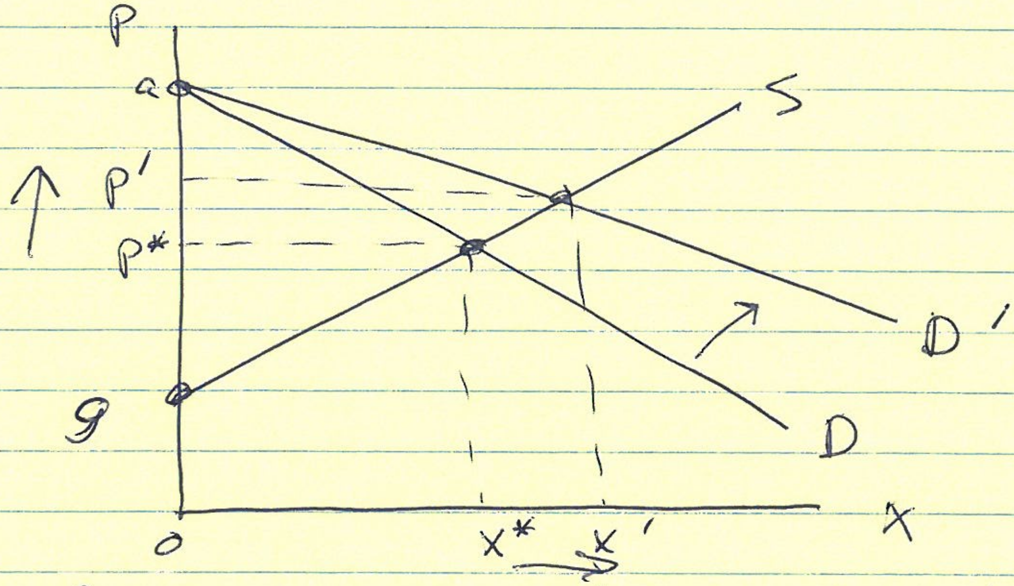
$S(p) = \dfrac{m(p-g)}{2h}$ for $p \ge g$; $S(p) = 0$ for $p \le g$.

Equate demand and supply: $D(p) = S(p)$

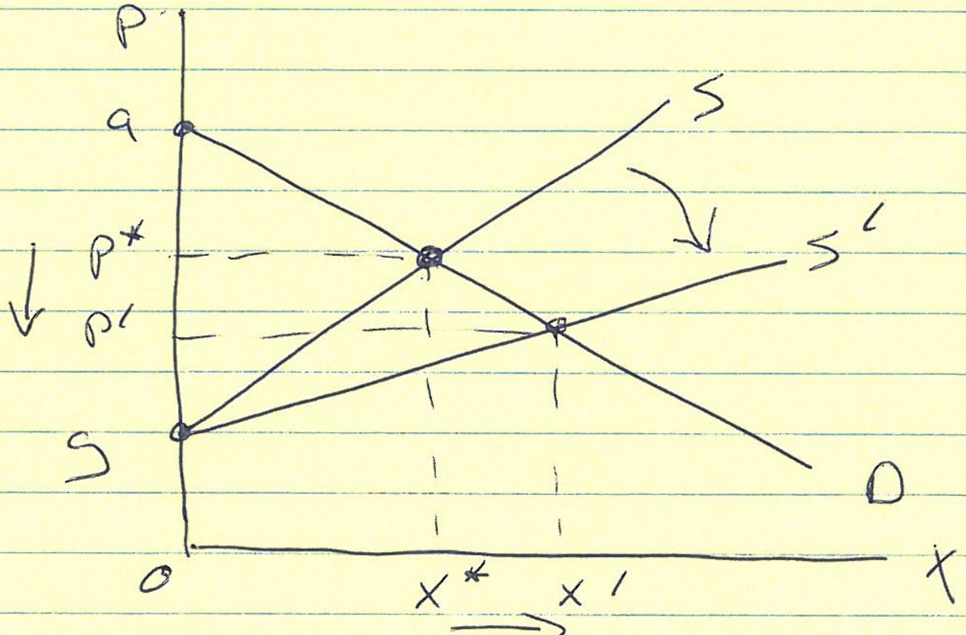
$\Rightarrow$

$$\frac{n(a-p)}{2b} = \frac{m(p-g)}{2h} \Rightarrow \boxed{p^* = \frac{nah + mbg}{nh + mb}}$$






If n increases while m is constant, the demand curve rotates around the vertical intercept at a to a curve like D' => both price and quantity increase




If m increases while n is constant, the supply curve rotates around the vertical intercept at g ~~to~~ a curve like S' => price falls, quantity rises.

5(b) To max the sum of the utilities, we need

$$\max \sum_i a x_i - \sum_i b x_i^2 + \sum_i y_i$$

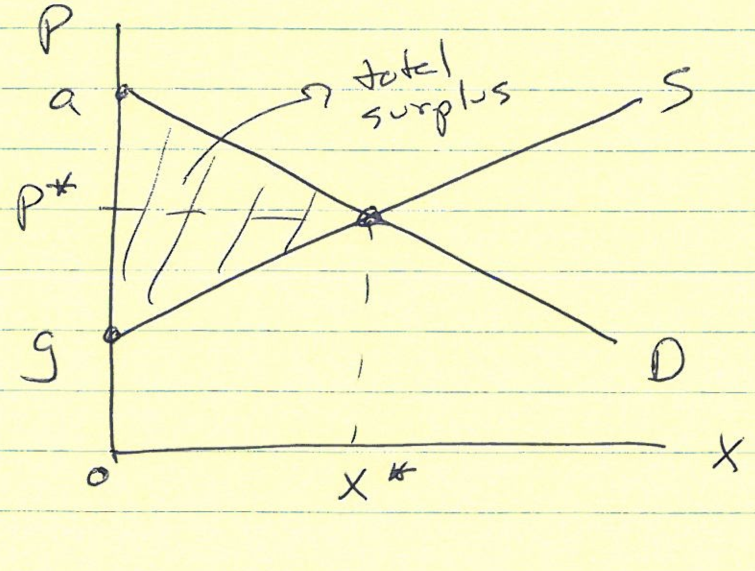
subject to $\sum_i x_i = \sum_j z_j$

and $\sum_i y_i = \sum_i w_i - \sum_j c_j(z_j)$ $\rightarrow [g z_j + h z_j^2]$

Eliminate $\sum_i y_i$ by substitution, set up a Lagrangean, and derive FOC to get

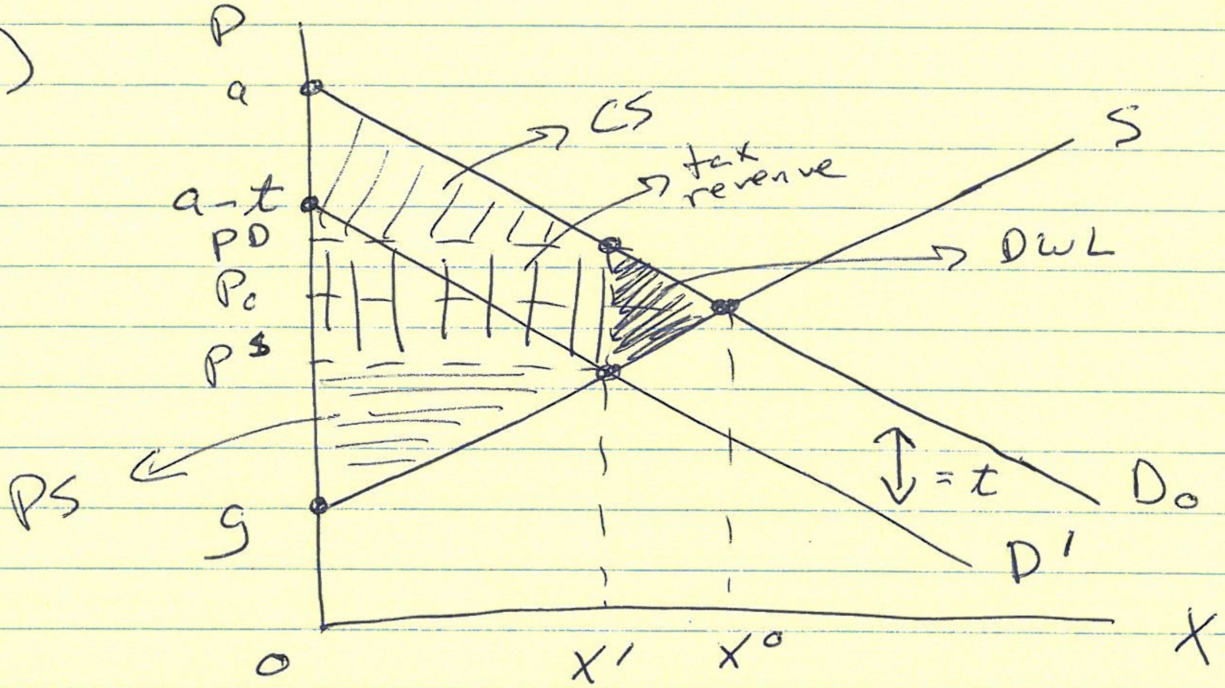
$$a - 2bx_i - \lambda = 0 \qquad i = 1..n$$
$$-g - 2hz_j + \lambda = 0 \qquad j = 1..m$$

The FOC are sufficient due to strict concavity of utility and strict convexity of cost.
Notice that these FOC are identical to the ones in part (a), and the constraints in this problem are identical to the market clearing ~~problems~~ conditions in part (a). Therefore any allocation that is a market equilibrium from part (a) must also maximize the sum of the utilities.




When we max the sum of the utilities, we are maximizing total surplus (the area below D and above S). The market equilibrium does this automatically (note $p^* = \lambda$).

5(c)




Because consumers pay the tax, we shift the demand curve down from $D_0$ to $D'$ in a parallel way. The old (pre-tax) equilibrium was $(P^c, X^0)$. The new (post-tax) equilibrium has a demand price $P^D$ and a supply price $P^S$ because the consumers pay $P^S$ to the firms and $P^D = P^S + t$ overall (they pay $t$ to the government). The new aggregate quantity is $X'$ which is less than $X^0$.

Total surplus is now consumer surplus + producer surplus + tax revenue. This is less than the previous total surplus by the area DWL = deadweight loss. This alters the answer from part (b) because total surplus is no longer maximized and thus the sum of the utilities is not maximized. The reduction in the sum of the utilities due to the tax is DWL.